Around a conjecture of Erdős on graph Ramsey numbers

L. Maherani[a], G.R. Omidi[a,b,1]

[a]Department of Mathematical Sciences, Isfahan University of Technology,
Isfahan, 84156-83111, Iran

[b]School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O.Box: 19395-5746, Tehran, Iran

l.maherani@math.iut.ac.ir, romidi@cc.iut.ac.ir

Abstract

For given graphs G_1 and G_2 the Ramsey number $R(G_1, G_2)$, is the smallest positive integer n such that each blue-red edge coloring of the complete graph K_n contains a blue copy of G_1 or a red copy of G_2. In 1983, Erdős conjectured that there is an absolute constant c such that $R(G) = R(G, G) \leq 2^{c\sqrt{m}}$ for any graph G with m edges and no isolated vertices. Recently this conjecture was proved by B. Sudakov. In this note, using the Sudakov's ideas we give an extension of his result and some interesting corollaries.

Keywords: Ramsey number, Erdős conjecture, Complete graph.

AMS Subject Classification: 05C55, 05D10.

1 Introduction

For given graphs G_1 and G_2 the *Ramsey number* $R(G_1, G_2)$, is the smallest positive integer n such that each blue-red edge coloring of the complete graph K_n contains a blue subgraph isomorphic to G_1 or a red subgraph isomorphic to G_2. We denote $R(G, G)$ by $R(G)$. The existence of such a positive integer is guaranteed by Ramsey's classical result [7]. Since 1970's, Ramsey theory has grown into one of the most active areas of research within combinatorics, overlapping variously with graph theory, number theory, geometry and logic. Probably the most complicated question in this field is the estimating the Ramsey number of complete graphs. A basic result of Erdős and Szekeres [6] implies the following theorem.

Theorem 1.1 ([6]) *For every positive integer n, $R(K_n) \leq 2^{2n}$.*

Using probabilistic methods, Erdős [3] obtained a lower bound for $R(K_n)$.

Theorem 1.2 ([3]) *For every positive integer $n > 2$, $R(K_n) \geq 2^{\frac{n}{2}}$.*

[1]This research was in part supported by a grant from IPM (No.90050049)

Over the last sixty years, there have been several improvements on these bounds. The best upper bound is obtained by Conlon [2]. A problem on Ramsey numbers of general graphs was posed by Erdős and Graham [5], who conjectured that among all graphs with $m = \binom{n}{2}$ edges and no isolated vertices, the complete graph on n vertices has the largest Ramsey number. Since the number of vertices in a complete graph with m edges is a constant multiple of $\sqrt{m}$ motivated by Theorem 1.1 and Theorem 1.2, Erdős conjectured [4] that there is a constant c such that for all graphs G with m edges and with no isolated vertices, $R(G) \leq 2^{c\sqrt{m}}$. The first result in this direction, proved by Alon, Krivelevich and Sudakov [1], is that $R(G) \leq 2^{c\sqrt{m}\log m}$. They also proved this conjecture for bipartite graphs. In 2011, Sudakov [8] gave a short and intelligent proof for this conjecture. In fact, he proved the following theorem.

Theorem 1.3 ([8]) *If G is a graph with m edges and with no isolated vertices, $R(G) \leq 2^{250\sqrt{m}}$.*

Our main results in this note are the following two theorems. The first theorem is an extension of Theorem 1.3 and will be proved by the same arguments in [8] and the second one can be obtained by the first main theorem and a result in [1] on the Ramsey number of a bounded maximum degree graph and a complete graph. We give their proofs in the last section. Through this note, the notations $\ln x$ and $\log x$ are the logarithms in the natural base e and 2, respectively.

Theorem 1.4 *Let G_i, $i = 1, 2$, be a graph of order n_i with no isolated vertices and m be a positive integer with $2^{\frac{106\sqrt{m}}{\log m}} \geq n - 27\sqrt{m}$, where $n = \max\{n_1, n_2\}$. Also suppose that for each $27 \leq \alpha \leq \frac{\log^3 m}{8}$ we have the following properties:*

I. There exists $U_i \subseteq V(G_i)$, $i = 1, 2$, so that $|U_i| \leq \alpha\sqrt{m}$ and $\Delta(G_i - U_i) \leq \frac{2\sqrt{m}}{\alpha}$,

II. There exists $V_i \subseteq V(G_i)$, $i = 1, 2$, so that $|V_i| \leq m^{\frac{3}{2}}$ and $\max\{R(G_1 - V_1, G_2), R(G_1, G_2 - V_2)\} \leq 2^{36\sqrt{m}}$.

Then $R(G_1, G_2) \leq 2^{250\sqrt{m}}$.

The join of two graphs G and H, denoted by $G + H$, is a graph with vertex set $V(G \cup H)$ and edge set $E(G \cup H) \cup \{uv | u \in G, v \in H\}$.

Theorem 1.5 *Let $G_1 = K_p$, $m \geq 27$, $p \leq 27\sqrt{m} + \frac{16\sqrt{m}}{\log^3 m}$ and $G_2 = K_l + H$ where $l \leq 27\sqrt{m}$, $\Delta(H) \leq \frac{16\sqrt{m}}{\log^3 m}$, $n(H) \leq 2^{\frac{106\sqrt{m}}{\log m}}$ and there is $S \subseteq V(H)$ with $|S| \leq \sqrt{m^3} - 27\sqrt{m}$ such that $\Delta(H - S) < \frac{\log m}{4}$. Then*

$$R(G_1, G_2) \leq 2^{250\sqrt{m}}.$$

Immediately we can obtained some corollaries using Theorems 1.4 and 1.5.

Corollary 1.6 *Suppose that $G_i, i = 1, 2$, is a graph with m_i edges and with no isolated vertices. If $m = \max\{m_1, m_2\}$, then $R(G_1, G_2) \leq 2^{250\sqrt{m}}$.*

Proof. Since G_i, $i = 1, 2$, has at most $2m$ vertices, then by the Theorem 1.1, $R(G_1, G_2) \leq R(K_{2m}, K_{2m}) \leq 2^{4m}$. If $m \leq 60^2$, then $4m \leq 250\sqrt{m}$ and so $R(G_1, G_2) \leq 2^{250\sqrt{m}}$. So we may assume that $m \geq 60^2$. Obviously $2^{\frac{106\sqrt{m}}{\log m}} \geq 2m - 27\sqrt{m} \geq n - 27\sqrt{m}$, where $n = \max\{n_1, n_2\}$ and n_i, $i = 1, 2$, is the number of vertices of G_i. Again clearly for $27 \leq \alpha \leq \frac{\log^3 m}{8}$, the maximum degree of a graph obtained by deleting $\alpha\sqrt{m}$ vertices of large degree of G_i is at most $\frac{2m}{\alpha\sqrt{m}} = \frac{2\sqrt{m}}{\alpha}$. On the other hand, $m^{\frac{3}{2}} \geq n$. Hence the assertion holds by Theorem 1.4. ■

Corollary 1.7 *Suppose G_i, $i = 1, 2$, is a graph of order n_i and $n = \max\{n_1, n_2\}$. If there exists a subset $U_i \subseteq V(G_i)$, $i = 1, 2$, with $|U_i| \leq 27\sqrt[3]{n}$ such that $\Delta(G_i - U_i) \leq 54\frac{\sqrt[3]{n}}{\log^3 n}$, Then $R(G_1, G_2) \leq 2^{250\sqrt[3]{n}}$.*

Proof. The assertion holds by putting $m = n^{\frac{2}{3}}$ in Theorem 1.4. ■

Using Theorem 1.5, we get the following corollaries.

Corollary 1.8 *Let $G_1 = K_p$, $m \geq 27$, $p \leq 27\sqrt{m} + \frac{16\sqrt{m}}{\log^3 m}$ and $G_2 = K_l + qK_1$, where $l \leq 27\sqrt{m}$ and $q \leq 2^{106\frac{\sqrt{m}}{\log m}}$. Then $R(G_1, G_2) \leq 2^{250\sqrt{m}}$.*

Corollary 1.9 *For positive integers p and q with $q \leq 2^{\frac{53p}{27\log\frac{p}{27}}}$, we have $R(K_p, K_{p,q}) \leq R(K_p, K_p + qK_1) \leq 2^{\frac{250}{27}p}$.*

2 Preliminaries

In this section, we present some lemmas which will be used in the proof of Theorem 1.4. Let $G = (V, E)$ be a graph and $U \subseteq V$. The induced subgraph of G on U is denoted by $G[U]$. The edge density of $G[U]$ is denoted by $d(U)$ and is defined by $d(U) = \frac{e(U)}{\binom{|U|}{2}}$, where $e(U)$ is the number of edges of $G[U]$. In an edge-coloring of K_n, we call an ordered pair (X, Y) of disjoint subsets of vertices monochromatic if all edges in $X \cup Y$ incident to each vertex in X have the same color.

Lemma 2.1 ([8]) *For all k and l, every blue-red edge coloring of K_N, contains a monochromatic pair (X, Y) with*

$$|Y| \geq \binom{k+l}{k}^{-1} N - k - l,$$

which is red and $|X| = k$ or is blue and $|X| = l$.

Lemma 2.2 ([8]) *Let $0 < \epsilon \leq \frac{1}{7}$ and let t and N be positive integers satisfying $t \geq \epsilon^{-1}$ and $N \geq t\epsilon^{-14\epsilon t}$. Then every blue-red edge coloring of $G = K_N$ in which edge density of the induced subgraph on the blue edges is at most ϵ contains a monochromatic pair (X, Y) with $|X| \geq t$ and $|Y| \geq \epsilon^{14\epsilon t}N$.*

Lemma 2.3 ([8]) *Let G be a graph with n vertices, maximum degree Δ and let $\epsilon \leq \frac{1}{8}$. If H has $N \geq \epsilon^{4\Delta\log\epsilon}n$ vertices and does not contain a copy of G, then it has a subset S with $|S| \geq \epsilon^{-4\Delta\log\epsilon}N$ and edge density $d(S) \leq \epsilon$.*

Lemma 2.4 *Let G_1 and G_2 be graphs with no isolated vertices of orders n_1 and n_2, respectively, and m be a positive integer such that $2^{\frac{106\sqrt{m}}{\log m}} \geq n - 27\sqrt{m}$, where $n = \max\{n_1, n_2\}$. Also suppose that for $27 \leq \alpha \leq \frac{\log^3 m}{8}$ there exists $U_i \subseteq V(G_i)$ with $|U_i| \leq \alpha\sqrt{m}$ such that $\Delta(G_i') \leq \frac{2\sqrt{m}}{\alpha}$, where $G_i' = G_i - U_i$. If a blue-red edge coloring of K_N has no blue copy of G_1 and no red copy of G_2 and it contains a monochromatic pair (X, Y) with $|X| \geq \alpha\sqrt{m}$ and $|Y| \geq 2^{125\alpha^{\frac{-1}{3}}\sqrt{m}}$, then it also contains a monochromatic pair (X', Y') with $|X'| \geq 2^{2\alpha^{\frac{1}{3}}}\sqrt{m}$ and $|Y'| \geq 2^{-120\alpha^{\frac{-1}{3}}\sqrt{m}}|Y|$.*

Proof. Consider a blue-red edge coloring of K_N with the properties in Lemma 2.4. Assume that the color of the monochromatic pair (X, Y) is blue. By the hypothesis, the induced subgraph on Y contains no blue G_1'. Otherwise, together with X, we get a blue G_1. Let $\epsilon = 2^{-3\alpha^{\frac{1}{3}}}$ and $t = 2^{2\alpha^{\frac{1}{3}}}\sqrt{m}$. Since $\alpha \leq \frac{\log^3 m}{8}$ we have $2^{\alpha^{\frac{1}{3}}} \leq \sqrt{m}$ and so $t \geq \epsilon^{-1}$. Also note that, since $27 \leq \alpha \leq \frac{\log^3 m}{8}$, we have $42\alpha^{\frac{1}{3}}2^{-\alpha^{\frac{1}{3}}} \leq 48\alpha^{\frac{-1}{3}}$

and $2^{5\alpha^{\frac{-1}{3}}\sqrt{m}} \geq 2^{10\frac{\sqrt{m}}{\log m}} \geq m^{\frac{3}{2}} \geq 2^{2\alpha^{\frac{1}{3}}}\sqrt{m} = t$. Applying Lemma 2.3 to the blue graph restricted to Y, we find a subset $S \subset Y$ with

$$\begin{aligned} |S| &\geq \epsilon^{-4\Delta(G_1')\log\epsilon}|Y| \geq (2^{-3\alpha^{\frac{1}{3}}})^{-4(2\alpha^{-1}\sqrt{m})(-3\alpha^{\frac{1}{3}})}|Y| \\ &= 2^{-72\alpha^{\frac{-1}{3}}\sqrt{m}}|Y| \geq 2^{53\alpha^{\frac{-1}{3}}\sqrt{m}} \geq 2^{\frac{106\sqrt{m}}{\log m}} \geq n - 27\sqrt{m}, \end{aligned}$$

such that the density of the induced subgraph on the blue edges in S is at most ϵ. Then the size of S satisfies

$$|S| \geq 2^{53\alpha^{\frac{-1}{3}}\sqrt{m}} \geq 2^{5\alpha^{\frac{-1}{3}}\sqrt{m}}2^{48\alpha^{\frac{-1}{3}}\sqrt{m}} \geq t2^{42\alpha^{\frac{1}{3}}2^{-\alpha^{\frac{1}{3}}}\sqrt{m}} = t\epsilon^{-14\epsilon t},$$

and we can apply Lemma 2.2 to S. So S contains a monochromatic pair (X', Y') with $|X'| \geq t$ and $|Y'| \geq \epsilon^{14\epsilon t}|S|$. To complete the proof, recall that $|S| \geq 2^{-72\alpha^{\frac{-1}{3}}\sqrt{m}}|Y|$ and therefore

$$|Y'| \geq \epsilon^{14\epsilon t}|S| \geq 2^{-48\alpha^{\frac{-1}{3}}\sqrt{m}}|S| \geq 2^{-120\alpha^{\frac{-1}{3}}\sqrt{m}}|Y|.$$

A similar argument, which we omit, can be used to finish the proof in the case when the color of the monochromatic pair (X, Y) is red. ■

3 The Proofs

In this section, we give the proofs for the main theorems.

The proof of Theorem 1.4. Assume that $N = 250\sqrt{m}$ and suppose for contradiction that there is a blue-red edge coloring of K_N with no blue copy of G_1 and no red copy of G_2. Since G_1 and G_2 have at most n vertices by Theorem 1.1, we have $R(G_1, G_2) \leq R(K_n) \leq 2^{2n}$. So we may assume that $n \geq 125\sqrt{m}$. Applying Lemma 2.1 with $k = l = 27\sqrt{m}$, we have a monochromatic pair (X_1, Y_1) with $|X_1| \geq 27\sqrt{m}$ and

$$|Y_1| \geq \binom{k+l}{k}^{-1} N - k - l \geq 4^{-27\sqrt{m}}N = 2^{196\sqrt{m}}.$$

Define $\alpha_1 = 27$ and $\alpha_{i+1} = 2^{2\alpha_i^{\frac{1}{3}}}$. It is easy to see $\alpha_{i+1} \geq (\frac{4}{3})^3\alpha_i$ for all i and therefore $\alpha_i^{-\frac{1}{3}} \leq \frac{1}{3}(\frac{3}{4})^{i-1}$. This implies that

$$\Sigma_{j=1}^{i}\alpha_j^{-\frac{1}{3}} \leq \frac{1}{3}\Sigma_{j=0}^{i-1}(\frac{3}{4})^j \leq \frac{1}{3}\Sigma_{j\geq 0}(\frac{3}{4})^j = \frac{4}{3}.$$

Since the blue-red edge coloring of K_N has no blue copy of G_1 and no red copy of G_2, we can repeatedly apply Lemma 2.4. After i iterations, we have a monochromatic pair (X_{i+1}, Y_{i+1}) with $|X_{i+1}| \geq \alpha_{i+1}\sqrt{m}$ and

$$|Y_{i+1}| \geq 2^{-120\alpha_i^{\frac{-1}{3}}\sqrt{m}}|Y_i| \geq 2^{-120\sqrt{m}\Sigma_{j=1}^{i}\alpha_j^{-\frac{1}{3}}}|Y_1| \geq 2^{-120\sqrt{m}(\frac{4}{3})}2^{196\sqrt{m}} = 2^{36\sqrt{m}}.$$

We continue iterations until the first index i such that $\alpha_i \geq \frac{\log^3 m}{8}$. Then for $\alpha = \frac{\log^3 m}{8}$ we have a monochromatic pair (X, Y), $|X| \geq \alpha\sqrt{m}$ and $|Y| \geq 2^{36\sqrt{m}} \geq 2^{125\alpha^{\frac{-1}{3}}\sqrt{m}}$. Then applying Lemma 2.4 one more time we obtain a monochromatic pair (X', Y') with $|X'| \geq 2^{2\alpha^{\frac{1}{3}}}\sqrt{m} = m^{\frac{3}{2}}$ and $|Y'| \geq 2^{36\sqrt{m}}$. First let (X', Y') be blue. By the second property in the hypothesis, since the induced subgraph on Y' contains no a red G_2, it contains a blue copy of $G_1 - V_1$ and clearly $X' \cup (G_1 - V_1)$ contains a blue copy of G_1, a contradiction. So we may assume that the monochromatic pair (X', Y') is red. Clearly the subgraph on Y' does not contain a blue G_1 and so it contains a red $G_2 - V_2$ and so $X' \cup (G_2 - V_2)$ contains a red copy of G_2, a contradiction. ■

To prove Theorem 1.5 we need the following result in [1].

Theorem 3.1 *([1]) Let H be a graph with h vertices and chromatic number $\chi(H) = k \geq 2$. Suppose that there is a proper k-coloring of H in which the degrees of all vertices, besides possibly those in the first color class, are at most r, where $1 \leq r(< h)$. Define $\alpha(k, r)$ to be 1 if $k > r$, and 0 otherwise. Then, for every integer $m > 1$,*

$$R(H, K_m) \leq (\frac{100m}{\ln m})^{\frac{(2r-k+2)(k-1)}{2}}(\ln m)^{\alpha(k,r)}h^r.$$

The proof of Theorem 1.5. Let $q = n(H - S)$, $k = \chi(H - S)$ and $r = \Delta(H - S)$. Since $m \geq 27$, then $p \leq 40\sqrt{m}$. If $r = 0$, we add an edge, say e, to $H - S$. By Theorem 3.1 we have

$$\begin{aligned}
R(H-S, G_1) &\leq R(H - S + e, G_1) \\
&< 100pq \leq 100(27\sqrt{m} + \frac{16\sqrt{m}}{\log^3 m})2^{106\frac{\sqrt{m}}{\log m}} \\
&< 4000\sqrt{m}2^{\frac{106\sqrt{m}}{\log m}} = 2^{\log 4000\sqrt{m}+106\frac{\sqrt{m}}{\log m}} \\
&\leq 2^{13+\frac{1}{2}\log m+106\frac{\sqrt{m}}{\log m}} < 2^{36\sqrt{m}}.
\end{aligned}$$

Then we can use Theorem 1.4. Now suppose that $r \geq 1$. Using Theorem 3.1, we get

$$\begin{aligned}
R(H-S, G_1) = R(H-S, K_p) &\leq (\frac{100p}{\ln p})^{\frac{(2r-k+2)(k-1)}{2}}(\ln p)(q)^r \\
&\leq (\frac{100p}{\ln p})^{\frac{r^2+r}{2}}(\ln p)(27\sqrt{m} + 2^{106\frac{\sqrt{m}}{\log m}} - \sqrt{m^3})^r \\
&\leq \frac{(100p)^{\frac{r^2+r}{2}}2^{106\frac{\sqrt{m}}{\log m}r}}{(\ln p)^{\frac{r^2+r-2}{2}}} \\
&\leq (4000\sqrt{m})^{\frac{r^2+r}{2}}2^{106\frac{\sqrt{m}}{\log m}r}y \\
&= 2^{(\frac{r^2+r}{2}\log 4000\sqrt{m}+106\frac{\sqrt{m}}{\log m}r)}y \\
&< 2^{36^{\sqrt{m}}}.
\end{aligned}$$

The last inequality holds, since

$$y = \frac{1}{(\ln p)^{\frac{r^2+r-2}{2}}} \leq 1,$$

and

$$\begin{aligned}
\frac{r^2+r}{2}\log 4000\sqrt{m} + 106\frac{\sqrt{m}}{\log m}r &= \frac{r^2+r}{2}(\log 4000 + \log\sqrt{m}) + 106\frac{\sqrt{m}}{\log m}r \\
&< 6(r^2+r) + \frac{r^2+r}{2}\log\sqrt{m} + 106\frac{\sqrt{m}}{\log m}r \\
&\leq 12(\log^2\sqrt[4]{m}) + \log^2\sqrt[4]{m}\log\sqrt{m} + 26.5\sqrt{m} \\
&= \frac{12}{16}\log^2 m + \frac{1}{32}\log^3 m + 26.5\sqrt{m} \\
&< 9.5\sqrt{m} + 26.5\sqrt{m} = 36\sqrt{m}.
\end{aligned}$$

Now we use Theorem 1.4 to finish the proof. ■

References

[1] N. Alon, M. Krivelevich, B. Sudakov, Turan numbers of bipartite graphs and related Ramsey-type questions, *Combin. Probab. Comput.* **12** (2003), 477-494.

[2] D. Conlon, A new upper bound for diagonal Ramsey numbers, *Ann. of Math.* **170** (2009), 941-960.

[3] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292-294.

[4] P. Erdős, On some problems in graph theory, combinatorial analysis and combinatorial number theory, *Graph Theory and Combinatorics*, Cambridge, 1983, *Academic Press, London*, 1984, 1-17.

[5] P. Erdős, R. Graham, On partition theorems for finite graphs, *in: Infinite and Finite Sets*, vol. I, Colloq., Keszthely, 1973, in: Colloq. Math. Soc. Janos Bolyai, vol. 10, North-Holland, Amsterdam, 1975, 515-527.

[6] P. Erdős, G. Szekeres, A Combinatorial problem in geometry, *Compos. Math.* **2** (1935), 463-470.

[7] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.*(2) 30 (1930), 264-286.

[8] B. Sudakov, A Conjecture of Erdős on graph Ramsey numbers, *Adv. Math.* **227** (2011), 601-609.